

From Einstein's Hole Argument to Dirac and Bergmann Observables.

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Abstract

The individuation of point-events and the Hamiltonian way of distinguishing gravitational from inertial effects in general relativity are discussed.

The fact that both particle physics and all the approaches to gravity make use of variational principles employing singular Lagrangians [1] has the consequence that the Euler-Lagrange equations cannot be put in normal form, some of them may be non independent equations (due to the contracted Bianchi identities) and a subset of the original configuration variables, the *gauge variables*, are left completely undetermined. This leads to the necessity of a division of the initial configuration variables of any model in two groups:

- i) the arbitrary *non determined gauge variables*;
- ii) the *gauge-invariant observables* with a deterministic evolution.

But this process is in conflict with locality, manifest Lorentz covariance, general covariance and, moreover, the configuration space manifestly covariant approach has no natural analytical tool to perform this separation.

Another non trivial aspect of the need of the division between gauge variables and deterministic observables is the connection of the latter with *measurable quantities*. Since, at least at the classical level, the electro-magnetic measurable quantities are the local electric and magnetic fields, we can extrapolate that the *non-local* radiation gauge observables, i.e. the transverse vector gauge potential and the transverse electric field, are also measurable. But in the case of the non-Abelian Yang-Mills gauge theories for the strong and weak interactions the connection between gauge invariant observables and measurable quantities is still poorly understood.

When we come to general relativity in Einstein formulation, these problems become both more complex and more basic. More complex because the Lie groups underlying the gauge

groups of particle physics are replaced by diffeomorphism groups¹, whose group manifold in large is poorly understood. More basic because now the action of the gauge group is not in an inner space of a field theory on a background space-time, but is an extension to tensors over space-time of the diffeomorphisms of the space-time itself. This reflects itself in the much more singular nature of Einstein's equations² with respect to Yang-Mills equations. This fact has the dramatic consequence to *destroy any physical individuality of the points of space-time* as evidenced by Einstein's *Hole Argument* [3] in the years (1913-16) of the genesis of the concept of general covariance. Only the idealization (*point-coincidence argument*) according to which all possible observations reduce to the intersections of the world-lines of observers, measuring instruments and measured physical objects, convinced Einstein to adopt general covariance and to abandon the physical objectivity of space-time coordinates. In Einstein's own words [3]:

"That the requirement of general covariance, *which takes away from space and time the last remnant of physical objectivity*, is a natural one, will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time. The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences".

At first sight it could seem from these words that Einstein simply equated general covariance with the unavoidable *arbitrariness of the choice of coordinates*, a fact that, in modern language, can be translated into *invariance under passive diffeomorphisms*. Actually the essence of the *point-coincidence argument* seems to be well in tune with the Machian epistemology Einstein shared at the time, in particular as regards the ontological privilege of "bodies" or "fields" versus "space".

The Hole Argument, after a long oblivion, was resurrected by Stachel [4] and then by Norton [5] and others as a basic problem [6] in our both ontological and physical understanding of *space-time* in general relativity, which is commonly thought to imply that space-time points have no *intrinsic physical meaning* due to the general covariance of Einstein's equations. This feature is implicitly described in standard modern textbooks by the statement that solutions to the Einstein's equations related by (*active*) diffeomorphisms have physically

¹Reparametrization invariant theories in Minkowski space-time for particles and strings and parametrized Minkowski theories for every isolated system [1,2] also have diffeomorphism groups as gauge groups.

²Four of them are not independent from the others due to the Bianchi identities, four are only restrictions on the initial data and only two combinations of Einstein's equations and their gradients depend on the *accelerations* (the second time derivatives of the metric tensor).

identical properties. Such kind of equivalence, which also embodies the modern understanding of Einstein's *Hole Argument*, has been named as *Leibniz equivalence* in the philosophical literature by Earman and Norton [7] and exploited to the effect of arguing against the *manifold substantivalism* (it is not possible to reconcile the need for determinism in classical physical laws and a *realistic* interpretation of the mathematical 4-manifold describing space-time) and in defense of the *relational* conception of space-time. However *Leibniz equivalence* is not and cannot be the last word about the *intrinsic physical properties* of space-time, well beyond the needs of the empirical grounding of the theory. In Ref. [8] (following Refs. [9]) an attempt is made to gain an intrinsic *dynamical characterization of space-time points in terms of the gravitational field itself*, besides and beyond the mathematical individuation furnished to them by the coordinates.

The basic mathematical concept that underlies the Hole Argument is the concept of *active diffeomorphism* and its consequent action on the tensor fields defined on a differentiable manifold. Our manifold will be the mathematical manifold M^4 , the first layer of the would-be physical space-time of general relativity. Consider a (geometrical or *active*) diffeomorphism ϕ which maps points of M^4 to points of M^4 : $\phi : p \rightarrow p' = \phi \cdot p$, and its tangent map ϕ^* which maps tensor fields $T \rightarrow \phi^* \cdot T$ in such a way that $[T](p) \rightarrow [\phi^* \cdot T](p) \equiv [T'](p)$. Then $[\phi^* \cdot T](p) = [T](\phi^{-1} \cdot p)$. It is seen that the transformed tensor field $\phi^* \cdot T$ is a *new* tensor field whose components in general will have at p values that are *different* from those of the components of T . On the other hand, the components of $\phi^* \cdot T$ have at p' - by construction - the same values that the components of the original tensor field T have at p : $T'(\phi \cdot p) = T(p)$ or $T'(p) = T(\phi^{-1} \cdot p)$. The new tensor field $\phi^* \cdot T$ is called the *drag-along* of T . Let us recall that there is another, non-geometrical - so-called *dual* - way of looking at the active diffeomorphisms, which, incidentally, is more or less the way in which Einstein himself formulated the original Hole Argument. This *duality* is based on the circumstance that in each region of M^4 covered by two or more charts there is a one-to-one correspondence between an *active* diffeomorphism and a specific coordinate transformation (or *passive* diffeomorphism). The coordinate transformation $\mathcal{T}_\phi : x(p) \rightarrow x'(p) = [\mathcal{T}_\phi x](p)$ which is *dual* to the active diffeomorphism ϕ is defined such that $[\mathcal{T}_\phi x](\phi \cdot p) = x(p)$. In its essence, this *duality* transfers the functional dependence of the new tensor field in the new coordinate system to the old system of coordinates. By analogy, the coordinates of the new system $[x']$ are said to have been *dragged-along* with the *active* diffeomorphism ϕ .

The right mathematical way of looking *passively* at the *active* diffeomorphisms has been studied by Bergmann and Komar [10]: they show that the group of active diffeomorphisms (${}_{\text{A}}\text{Diff } M^4$) can be described by a non-normal sub-group of a general group Q of *passive dynamical symmetries* of Einstein's equations [they correspond to generalized metric-dependent coordinate transformations $x'^\mu = f^\mu(x, g)$]. Q also contains two other non-normal sub-groups: the passive diffeomorphisms (or coordinate transformations, ${}_{\text{P}}\text{Diff } M^4$) and the set of those (either active or passive) diffeomorphisms which are projectable to phase space Q_{can} (they are interpretable as Hamiltonian gauge transformations generated by the first class constraints). Since passive diffeomorphisms play the role of Lagrangian gauge transformations, *a complete Lagrangian gauge fixing amounts to a definite choice of the coordinates on M^4* , a choice which, on the other hand, is necessary in order to explicitly solve the Einstein partial differential equations. In modern terminology, general covariance implies that

a physical solution of Einstein's equations properly corresponds to a *4-geometry*, namely the equivalence class of all the 4-metric tensors, solutions of the equations, written in all possible 4-coordinate systems. This equivalence class is usually represented by the quotient ${}^4\text{Geom} = {}^4\text{Riem}/_P\text{Diff } M^4$, where ${}^4\text{Riem}$ denotes the space of metric tensors solutions of Einstein's equations. Then, any two *inequivalent* Einstein space-times are different 4-geometries. As discussed in Ref. [8] Leibniz equivalence of metric tensors g means that an *Einstein (or on-shell, or dynamical) gravitational field* is an equivalence class of solutions of Einstein's equation *modulo* the dynamical symmetry transformations of $_A\text{Diff } M^4$. We also have ${}^4\text{Geom} = {}^4\text{Riem}/_P\text{Diff } M^4 = {}^4\text{Riem}/Q = {}^4\text{Riem}/_A\text{Diff } M^4 = {}^4\text{Riem}/Q_{can}$. It is clear that a parametrization of the 4-geometries should be grounded on the two independent dynamical degrees of freedom of the gravitational field. At the Hamiltonian level the canonical reduction is done *off-shell* (i.e. not on the solution of Einstein's equations): an *off-shell gravitational field* is an equivalence class under Hamiltonian gauge transformations containing many different 4-geometries and only the restriction to the solutions of Einstein's equations identifies a unique on-shell 4-geometry.

Now, the Hole Argument, in its modern version, runs as follows. Consider a general-relativistic space-time, as specified by the four-dimensional mathematical manifold M^4 and by a metric tensor field g which *represents at the same time the chrono-geometrical and causal structure of space-time and the potential for the gravitational field*. The metric g is a solution of the generally-covariant Einstein equations. If any non-gravitational physical fields are present, they are represented by tensor fields that are also dynamical fields, and that appear as sources in the Einstein equations. Assume now that M^4 contains a *Hole* \mathcal{H} : that is, an open region where all the non-gravitational fields are zero. On M^4 we can prescribe an *active* diffeomorphism ϕ that re-maps the points inside \mathcal{H} , but blends smoothly into the identity map outside \mathcal{H} and on the boundary. Now, *just because Einstein's equations are generally covariant* so that they can be written down as *geometrical relations*, if g is one of their solutions, so is the *drag-along* field $g' = \phi^* \cdot g$. By construction, for any point $p \in \mathcal{H}$ we have (geometrically) $g'(\phi \cdot p) = g(p)$, but of course $g'(p) \neq g(p)$ (also geometrically). Now, what is the correct interpretation of the new field g' ? Clearly, the transformation entails an *active redistribution of the metric over the points of the manifold*, so the crucial question is whether, to what extent, and how the points of the manifold are primarily *individuated*.

In the mathematical literature about topological spaces, it is always implicitly assumed that the entities of the set can be distinguished and considered separately (provided the Hausdorff conditions are satisfied), otherwise one could not even talk about point mappings or homeomorphisms. It is well known, however, that the points of a homogeneous space cannot have any intrinsic individuality³. There is only one way to individuate points at the mathematical level: namely by *coordinatization*, a procedure that transfers the individuality of 4-tuples of real numbers to the elements of the topological set. Precisely, one introduces by convention *a standard coordinate system* for the *primary individuation* of the points (like

³As Hermann Weyl [11] puts it: "There is no distinguishing objective property by which one could tell apart one point from all others in a homogeneous space: at this level, fixation of a point is possible only by a *demonstrative act* as indicated by terms like *this* and *there*."

the choice of *standards* in metrology). Then, one can get as many different *names*, for what we consider the same primary individuation, as the coordinate charts containing the point in the chosen atlas of the manifold. Therefore, all the relevant transformations operated on the manifold M^4 (including *active* diffeomorphisms which map points to points), even if viewed in purely geometrical terms, *must* be realizable in terms of coordinate transformations.

If one now thinks of the (mathematically individuated) points of \mathcal{H} as also *physically individuated* spatio-temporal events even before the metric is defined, then g and g' must be regarded as *physically distinct* solutions of the Einstein equations (after all, as already noted, $g'(p) \neq g(p)$ at the *same* point p). This, however, is a devastating conclusion for the causality of the theory, because it implies that, even after we completely specify a physical solution for the gravitational and non-gravitational fields outside the Hole - for example, on a Cauchy surface for the initial value problem - we are still *unable to predict uniquely the physical solution within the Hole*. As said the escape from the (mathematical) strictures of the Hole Argument, is to *deny that diffeomorphically related mathematical solutions represent physically distinct solutions*. With this assumption, *an entire equivalence class of diffeomorphically related mathematical solutions represents only one physical solution*.

It is seen at this point that the conceptual content of general covariance is far more deeper than the simple invariance under arbitrary changes of coordinates. Stachel [12,13] has given a very enlightening analysis of the meaning of general covariance and of its relations with the Hole Argument. He stresses that asserting that g and $\phi^* \cdot g$ represent *one and the same gravitational field* is to imply that *the mathematical individuation of the points of the differentiable manifold by their coordinates has no physical content until a metric tensor is specified*. In particular, coordinates lose any *physical significance whatsoever* [5]. Furthermore, as Stachel emphasizes, if g and $\phi^* \cdot g$ must represent the same gravitational field, they cannot be physically distinguishable in any way. So when we act on g with an active diffeomorphisms to create the drag-along field $\phi^* \cdot g$, *no element of physical significance* can be left behind: in particular, nothing that could identify a point p of the manifold as the *same* point of space-time for both g and $\phi^* \cdot g$. Instead, when p is mapped onto $p' = \phi \cdot p$, it *brings over its identity*, as specified by $g'(p') = g(p)$. A further important point made by Stachel is that simply because a theory has generally covariant equations, it does not follow that the points of the underlying manifold must lack any kind of physical individuation. Indeed, what really matters is that there can be no *non-dynamical individuating field* that is specified *independently* of the dynamical fields, and in particular independently of the metric. If this was the case, a *relative* drag-along of the metric with respect to the (supposedly) individuating field would be physically significant and would generate an inescapable Hole problem. Thus, the absence of any non-dynamical individuating field, as well as of any dynamical individuating field independent of the metric, is the crucial feature of the purely gravitational solutions of general relativity as well as of the very *concept of general covariance*. In the case of general relativity there is *no non-dynamical individuating field* like the distribution of rods and clocks in Minkowsky space-time, that can be specified independently of the dynamical fields, in particular independently of the metric. This conclusion led Stachel to the conviction that space-time points *must* be *physically individuated before* space-time itself acquires a physical bearing, and that the metric itself plays the privileged role of *individuating field*: a necessarily *unique role* in the case of space-time *without matter*. More precisely, Stachel claimed that this individuating role should be implemented by

four invariant functionals of the metric, already considered by Bergmann and Komar [14]. However, he did not follow up on his suggestion.

It is essential to realize that the *Hole Argument* is *inextricably entangled* with the initial value problem. Most authors have implicitly adopted the Lagrangian approach, where the Cauchy problem is in-tractable because of the non-hyperbolic nature of Einstein's equations (see Ref. [15] for an updated review). The constrained Hamiltonian approach is just the *only* proper way to analyze the initial value problem of that theory and to find the *deterministically predictable observables* of general relativity. It is not by chance that the modern treatment of the initial value problem within the Lagrangian configurational approach [15] must in fact mimic the Hamiltonian methods. Only in the Hamiltonian approach can we isolate the *gauge variables*, which carry the descriptive arbitrariness of the theory, from the *Dirac observables* (DO), which are gauge invariant quantities providing a coordinatization of the reduced phase space of general relativity, and are subjected to hyperbolic (and therefore "causal" in the customary sense) evolution equations. In physics the *Hole Argument* is considered an aspect of the fact that also Einstein's theory is interpreted as a gauge theory. The Leibnitz equivalence is nothing else than the selection of the gauge invariant observables of the theory. But now, differently from Yang-Mills theories, the physical interpretation of the underlying mathematical 4-manifold is lost, and this suggests that a different interpretation of the gauge variables of generally covariant theories with respect to Yang-Mills theories is needed.

As already said the manifestly covariant configuration space approach has no natural tool to make a clean separation between gauge variables and a basis of gauge invariant (hopefully measurable) observables. Instead, at least locally, the Hamiltonian formulation has natural tools for it, namely the Shanmugadhasan canonical transformations [16]. The singular Lagrangians of particle physics and general relativity imply the use of Dirac-Bergmann theory [17,1] of Hamiltonian constraints and only the constraint sub-manifold of phase space is relevant for physics. Let us consider a finite-dimensional system with configuration space Q with global coordinates q^i , $i = 1, \dots, N$ described by a singular Lagrangian $L(q, \dot{q})$ [$\dot{q}^i(\tau) = d q^i(\tau)/d\tau$]. Let the Dirac algorithm produce the following general pattern:

- i) $m < N$ first class constraints $\phi_\alpha(q, p) \approx 0$, of which the first $m_1 \leq m$ are primary, with the property that the Poisson brackets of any two of them satisfies $\{\phi_\alpha(q, p), \phi_\beta(q, p)\} = C_{\alpha\beta\gamma}(q, p) \phi_\gamma(q, p) \approx 0$;
- ii) $2n$ second class constraints, corresponding to pairs of canonical variables which can be eliminated by going to Dirac brackets;
- iii) a Dirac Hamiltonian $H_D = H_c + \sum_{\alpha=m_1+1}^m r_\alpha(q, p) \phi_\alpha(q, p) + \sum_{\alpha=1}^{m_1} \lambda_\alpha(\tau) \phi_\alpha(q, p)$, where the $\lambda_\alpha(\tau)$'s are arbitrary functions of time, named *Dirac multipliers*, associated only with the *primary* first class constraints ⁴. In phase space there will be as many arbitrary

⁴The use of the first half of Hamilton equations, $\dot{q}^i = \{q^i, H_D\}$, shows that the Dirac multipliers are those primary velocity functions ($g_\alpha(q, \dot{q}) = \lambda_\alpha(\tau)$ on the solutions of Hamilton equations) not determined by the singular Euler-Lagrange equations. It can be shown that this arbitrariness implies that also the secondary velocity functions $r_\alpha(q, p) = \tilde{r}_\alpha(q, \dot{q})$, $\alpha = m_1 + 1, \dots, m$, in front of

Hamiltonian gauge variables as first class constraints: they determine a coordinatization of the *gauge orbits* inside the constraint sub-manifold. The first class constraints are the generators of the Hamiltonian gauge transformations under which the theory is invariant and a gauge orbit is an equivalence class of all those configurations which are connected by gauge transformations (Leibnitz equivalence). The $2(N - m - n)$ -dimensional reduced phase space is obtained by eliminating the second class constraints with Dirac brackets and by going to the quotient with respect to the gauge orbits, or equivalently by adding as many gauge fixing constraints as first class ones so to obtain $2m$ second class constraints.

At least locally on the constraint sub-manifold the family of *Shanmugadhasan canonical transformations* $q^i, p_i \mapsto Q^\alpha, P_\alpha \approx 0, \bar{Q}^\beta \approx 0, \bar{P}_\beta \approx 0, Q^A, P_A, \alpha = 1, \dots, m, \beta = 1, \dots, n$, allows

- i) to Abelianize the first class constraints, so that locally the constraint submanifold is identified by the vanishing of a subset of the new momenta $P_\alpha \approx 0$;
- ii) to identify the associated Abelianized gauge variables Q^α as coordinates parametrizing the gauge orbits;
- iii) to replace the second class constraints with pairs of canonical variables $\bar{Q}^\beta \approx 0, \bar{P}_\beta \approx 0$;

iv) to identify a canonical basis of *gauge invariant Dirac observables* with a deterministic evolution determined only by the gauge invariant canonical part H_c of the Dirac Hamiltonian.

This is the tool of the Hamiltonian formalism, lacking in the configuration space approach, which allows to make the division between arbitrary gauge variables and deterministic gauge invariant observables. Since the (in general non local) Dirac observables give a coordinatization of the classical reduced phase space, it will depend on its topological properties whether a given system with constraints admits a sub-family of Shanmugadhasan canonical transformations globally defined. When this happens the system admits *preferred* global separations between gauge and observable degrees of freedom.

In ADM canonical gravity [18] ⁵ the ten components ${}^4g_{\mu\nu}$ of the 4-metric tensor are replaced by the following configuration variables: the *lapse* $N(\tau, \vec{\sigma})$ and *shift* $N_r(\tau, \vec{\sigma})$ functions and the six components of the *3-metric tensor* on Σ_τ , ${}^3g_{rs}(\tau, \vec{\sigma})$. Einstein's equations are then recovered as the Euler-Lagrange equations of the ADM action $S_{ADM} = -ek \int_{\Delta\tau} d\tau \int d^3\sigma \{ \sqrt{\gamma} N [{}^3R + {}^3K_{rs} {}^3K^{rs} - ({}^3K)^2] \}(\tau, \vec{\sigma})$, which differs from Einstein-Hilbert action by a suitable surface term. Here $k = \frac{c^3}{16\pi G}$, ${}^3K_{rs}$ is the extrinsic curvature of Σ_τ , 3K its trace, and 3R the 3-curvature scalar. Besides the ten configuration variables listed above, the ADM phase space is *coordinatized* by ten canonical momenta $\tilde{\pi}^N(\tau, \vec{\sigma}), \tilde{\pi}_{\vec{N}}^r(\tau, \vec{\sigma})$,

the secondary (and higher) first class constraints in H_D , are not determined by the Euler-Lagrange equations. Therefore each first class constraint has either a configuration or a generalized velocity as an arbitrary partner.

⁵The existence of a mathematical 4-manifold, the space-time M^4 , admitting 3+1 splittings with space-like leaves $\Sigma_\tau \approx R^3$ is assumed. All fields (also matter fields when present) depend on Σ_τ -adapted coordinates $(\tau, \vec{\sigma})$ for M^4 .

${}^3\tilde{\Pi}^{rs}(\tau, \vec{\sigma})$ ⁶. Such canonical variables, however, are not independent since they are restricted to the *constraint sub-manifold* by the eight *first class* constraints

$$\begin{aligned}\tilde{\pi}^N(\tau, \vec{\sigma}) &\approx 0, & \tilde{\pi}_{\vec{N}}^r(\tau, \vec{\sigma}) &\approx 0, \\ \tilde{\mathcal{H}}(\tau, \vec{\sigma}) &= \epsilon [k\sqrt{\gamma} {}^3R - \frac{1}{2k\sqrt{\gamma}} {}^3G_{rsuv} {}^3\tilde{\Pi}^{rs} {}^3\tilde{\Pi}^{uv}] (\tau, \vec{\sigma}) \approx 0, \\ {}^3\tilde{\mathcal{H}}^r(\tau, \vec{\sigma}) &= -2 {}^3\tilde{\Pi}^{rs}|_s(\tau, \vec{\sigma}) = -2[\partial_s {}^3\tilde{\Pi}^{rs} + {}^3\Gamma_{su}^r {}^3\tilde{\Pi}^{su}] (\tau, \vec{\sigma}) \approx 0.\end{aligned}\quad (0.1)$$

While the first four are *primary* constraints, the remaining four are the super-hamiltonian and super-momentum *secondary* constraints arising from the requirement that the primary constraints be constant in τ . More precisely, this requirement guarantees that, once we have chosen the initial data inside the constraint sub-manifold corresponding to a given initial Cauchy surface Σ_{τ_o} , the time evolution does not take them out of the constraint sub-manifolds for $\tau > \tau_o$. The eight infinitesimal off-shell Hamiltonian gauge transformations, generated by the first class constraints (0.1), have the following interpretation [19]:

- i) those generated by the four primary constraints modify the lapse and shift functions: these in turn determine how densely the space-like hyper-surfaces Σ_τ are distributed in space-time and also the conventions to be made on each Σ_τ about simultaneity (the choice of clocks synchronization) and gravito-magnetism;
- ii) those generated by the three super-momentum constraints induce a transition on Σ_τ from a given 3-coordinate system to another one;
- iii) that generated by the super-hamiltonian constraint induces a transition from a given 3+1 splitting of M^4 to another one, by operating normal deformations of the space-like hyper-surfaces⁷.

⁶As shown in Ref. [19], a consistent treatment of the boundary conditions at spatial infinity requires the explicit separation of the *asymptotic* part of the lapse and shift functions from their *bulk* part: $N(\tau, \vec{\sigma}) = N_{(as)}(\tau, \vec{\sigma}) + n(\tau, \vec{\sigma})$, $N_r(\tau, \vec{\sigma}) = N_{(as)r}(\tau, \vec{\sigma}) + n_r(\tau, \vec{\sigma})$, with n and n_r tending to zero at spatial infinity in a direction-independent way. On the contrary, $N_{(as)}(\tau, \vec{\sigma}) = -\lambda_\tau(\tau) - \frac{1}{2}\lambda_{\tau u}(\tau)\sigma^u$ and $N_{(as)r}(\tau, \vec{\sigma}) = -\lambda_r(\tau) - \frac{1}{2}\lambda_{ru}(\tau)\sigma^u$. The *Christodoulou-Klainermann space-times* [20], with their *rest-frame* condition of zero ADM 3-momentum and absence of super-translations, are singled out by these considerations. The allowed foliations of these space-times tend asymptotically to Minkowski hyper-planes in a direction-independent way and are asymptotically orthogonal to the ADM four-momentum. They have $N_{(as)}(\tau, \vec{\sigma}) = \epsilon$, $N_{(as)r}(\tau, \vec{\sigma}) = 0$. Therefore, in these space-times there are *asymptotic inertial time-like observers* (the *fixed stars* or the *CMB rest frame*) and the global mathematical time labeling the Cauchy surfaces can be identified with their rest time. For the sake of simplicity these aspects of the theory will be ignored, with the caveat that the canonical pairs $N, \tilde{\pi}^N, N_r, \tilde{\pi}_{\vec{N}}^r$ should be always replaced by the pairs $n, \tilde{\pi}^n, n_r, \tilde{\pi}_{\vec{n}}^r$.

⁷Note that in *compact* space-times the super-hamiltonian constraint is usually interpreted as generator of the evolution in some *internal time*, either like York's internal *extrinsic* time or like Misner's internal *intrinsic* time. Here instead the super-hamiltonian constraint is the generator of

The evolution in τ is ruled by the Hamilton-Dirac Hamiltonian

$$H_{(D)ADM} = \int d^3\sigma \left[N \tilde{\mathcal{H}} + N_r {}^3\tilde{\mathcal{H}}^r + \lambda_N \tilde{\pi}^N + \lambda_{\vec{N}}^{\vec{N}} \tilde{\pi}_{\vec{N}}^r \right] (\tau, \vec{\sigma}) \approx 0, \quad (0.2)$$

where $\lambda_N(\tau, \vec{\sigma})$ and $\lambda_{\vec{N}}^{\vec{N}}(\tau, \vec{\sigma})$ are *arbitrary Dirac multipliers* in front of the primary constraints⁸. This is just the Hamiltonian counterpart of the so-called "*indeterminism*" surfacing in the Hole Argument. The resulting hyperbolic system of Hamilton-Dirac equations has the same solutions of the non-hyperbolic system of (Lagrangian) Einstein's equations with the same boundary conditions.

The off-shell freedom corresponding to the eight independent types of Hamiltonian gauge transformations is reduced on-shell to four types like in the case of $PDiff M^4$: precisely the transformations in $[Q_{can} \cap PDiff M^4]$. At the off-shell level, this property is manifest by the circumstance that the original Dirac Hamiltonian contains *only* 4 arbitrary Dirac multipliers and that the *correct gauge-fixing procedure* [19] starts by giving *only* the four gauge fixing constraints for the secondary constraints. The requirement of time constancy then generates the four gauge fixing constraints to the primary constraints, while time constancy of such secondary gauge fixings leads to the determination of the four Dirac multipliers. Since the original constraints plus the above eight gauge fixing constraints form a second class set, it is possible to introduce the associated *Dirac brackets* and conclude the canonical reduction by realizing an off-shell reduced phase space. Of course, once a *completely fixed Hamiltonian gauge* is reached, general covariance is completely broken. Note that a completely fixed Hamiltonian gauge *on-shell* is equivalent to a *definite choice of the space-time 4-coordinates* on M^4 within the Lagrangian viewpoint.

In order to visualize the meaning of the various types of degrees of freedom one needs a determination of a *Shanmugadhasan canonical basis* [16] of metric gravity [19] having the following structure ($\bar{a} = 1, 2$ are non-tensorial indices of the DO⁹ $r_{\bar{a}}, \pi_{\bar{a}}$) with

| | | | | | | | | | | |
|---------------------------|-------------------------------------|----------------|--|--|---------------------------|-------------------------------------|-------------------------------------|-------------------------------------|-----------------------------|------------------|
| N | N_r | ${}^3g_{rs}$ | | | N | N_r | ξ^r | ϕ | $r_{\bar{a}}$ | |
| $\tilde{\pi}^N \approx 0$ | $\tilde{\pi}_{\vec{N}}^r \approx 0$ | ${}^3\Pi^{rs}$ | | | $\tilde{\pi}^N \approx 0$ | $\tilde{\pi}_{\vec{N}}^r \approx 0$ | $\tilde{\pi}_r^{\vec{H}} \approx 0$ | π_ϕ | $\pi_{\bar{a}}$ | |
| | | | | | \longrightarrow | N | N_r | ξ^r | $Q_{\mathcal{H}} \approx 0$ | $r'_{\bar{a}}$ |
| | | | | | \longrightarrow | $\tilde{\pi}^N \approx 0$ | $\tilde{\pi}_{\vec{N}}^r \approx 0$ | $\tilde{\pi}_r^{\vec{H}} \approx 0$ | $\Pi_{\mathcal{H}}$ | $\pi'_{\bar{a}}$ |

(0.3)

those Hamiltonian gauge transformations which imply that the description is independent of the choice of the allowed 3+1 splitting of space-time: *this is the correct answer to the criticisms raised against the phase space approach on the basis of its lack of manifest covariance*.

⁸These are four *velocity functions* (gradients of the metric tensor) which are not determined by Einstein's equations. As shown in Ref. [19], the correct treatment of the boundary conditions leads to rewrite Eq.(0.2) in terms of n and n_r , which are the arbitrary secondary velocity functions.

⁹The DO are in general neither tensors nor invariants under space-time diffeomorphisms. Therefore their (unknown) functional dependence on the original variables changes (off-shell) with the gauge and, therefore, (on-shell) with the 4-coordinate system.

It is seen that we need a sequence of two canonical transformations.

a) The first one replaces seven first-class constraints with as many Abelian momenta (ξ^r are the gauge parameters of the passive 3-diffeomorphisms generated by the super-momentum constraints) and introduces the conformal factor ϕ of the 3-metric as the configuration variable to be determined by the super-hamiltonian constraint¹⁰. Note that the final gauge variable, namely the momentum π_ϕ conjugate to the conformal factor, is the only gauge variable of momentum type: it plays the role of a *time* variable, so that the Lorentz signature of space-time is made manifest by the Shanmugadhasan transformation in the set of gauge variables $(\pi_\phi; \xi^r)$. More precisely, the first canonical transformation should be called a *quasi-Shanmugadhasan* transformation, because nobody has succeeded so far in Abelianizing the super-hamiltonian constraint. Note furthermore that this transformation is a *point* canonical transformation.

b) The second canonical transformation would be instead a *complete Shanmugadhasan* transformation, where $Q_{\mathcal{H}}(\tau, \vec{\sigma}) \approx 0$ would denote the Abelianization of the super-hamiltonian constraint¹¹. The variables $N, N_r, \xi^r, \Pi_{\mathcal{H}}$ are the final *Abelianized Hamiltonian gauge variables* and $r'_{\bar{a}}, \pi'_{\bar{a}}$ the final DO. In absence of explicit solutions of the Lichnerowicz equation, the best we can do is to construct the *quasi-Shanmugadhasan* transformation. On the other hand, such transformation has the remarkable property that, in the *special gauges* with $\pi_\phi(\tau, \vec{\sigma}) \approx 0$, the variables $r_{\bar{a}}, \pi_{\bar{a}}$ form a canonical basis of off-shell DO for the gravitational field *even if* the solution of the Lichnerowicz equation is not known.

The four gauge fixings to the secondary constraints, when written in the quasi-Shanmugadhasan canonical basis, have the following meaning:

i) the three gauge fixings for the parameters ξ^r of the spatial passive diffeomorphisms generated by the super-momentum constraints correspond to the choice of a system of 3-

¹⁰Recall that the *strong* ADM energy is the flux through the surface at spatial infinity of a function of the 3-metric only, and it is weakly equal to the *weak* ADM energy (volume form) which contains all the dependence on the ADM momenta. This implies [19] that the super-hamiltonian constraint must be interpreted as the equation (*Lichnerowicz equation*) that uniquely determines the *conformal factor* $\phi = (\det{}^3 g)^{1/12}$ of the 3-metric as a functional of the other variables. This means that the associated gauge variable is the *canonical momentum* π_ϕ *conjugate to the conformal factor*: this latter carries information about the extrinsic curvature of Σ_τ . It is just this variable, and *not* York's time, which parametrizes the *normal* deformation of the embeddable space-like hyper-surfaces Σ_τ .

¹¹If $\tilde{\phi}[r_{\bar{a}}, \pi_{\bar{a}}, \xi^r, \pi_\phi]$ is the solution of the Lichnerowicz equation, then $Q_{\mathcal{H}} = \phi - \tilde{\phi} \approx 0$. Other forms of this canonical transformation should correspond to the extension of the York map [21] to asymptotically flat space-times: in this case the momentum conjugate to the conformal factor would be just York time and one could add the maximal slicing condition as a gauge fixing. Again, however, nobody has been able so far to build a York map explicitly.

coordinates on Σ_τ ¹². The time constancy of these gauge fixings generates the gauge fixings for the shift functions N_r while the time constancy of the latter leads to the fixation of the Dirac multipliers $\lambda_r^{\vec{N}}$;

ii) the gauge fixing to the super-hamiltonian constraint determines π_ϕ : it is a fixation of the form of Σ_τ and amounts to the choice of one particular 3+1 splitting of M^4 . Since the time constancy of the gauge fixing on π_ϕ determines the gauge fixing for the lapse function N (and then of the Dirac multiplier λ_N), it follows a connection with the choice of the standard of local proper time.

All this entails that, after such a fixation of the gauge G , the functional form of the DO in terms of the original variables becomes gauge-dependent. At this point it is convenient to denote them as $r_{\vec{a}}^G$, $\pi_{\vec{a}}^G$. Since the Shanmugadhasan canonical transformation is a *highly non-local* transformation and it is not known how to build a global atlas of coordinate charts for the group manifold of diffeomorphism groups, it is not known either how to express the ξ^r 's, π_ϕ and the DO in terms of the original ADM canonical variables.¹³. In conclusion, a representative of a *Hamiltonian kinematic or off-shell gravitational field*, in a given gauge equivalence class, is parametrized by $r_{\vec{a}}$, $\pi_{\vec{a}}$ and is an element of a *conformal gauge orbit* (it contains all the 3-metrics in a conformal 3-geometry) spanned by the gauge variables ξ^r , π_ϕ , N , N_r . Therefore, according to the gauge interpretation based on constraint theory, a *Hamiltonian kinematic or off-shell gravitational field* is an equivalence class of 4-metrics modulo the Hamiltonian group of gauge transformations, which contains a well defined conformal 3-geometry.

The previous discussion applies to a class of globally hyperbolic, topologically trivial, non-compact (asymptotically flat at spatial infinity) space-times of the type of Christodoulou-Klainermann ones [20]. In them we have [19,8]:

1) The imposition of suitable boundary conditions on the fields and the gauge transformations of canonical ADM metric gravity eliminates the super-translations and reduces the asymptotic symmetries at spatial infinity to the asymptotic ADM Poincaré group. The asymptotic implementation of Poincaré group makes possible the general-relativistic definition of angular momentum and the matching of general relativity with particle physics.

¹²Since the diffeomorphism group has no canonical identity, this gauge fixing has to be done in the following way. One chooses a 3-coordinate system by choosing a parametrization of the six components ${}^3g_{rs}(\tau, \vec{\sigma})$ of the 3-metric in terms of *only three* independent functions. This amounts to fix the three functional degrees of freedom associated with the diffeomorphism parameters $\xi^r(\tau, \vec{\sigma})$. For instance, a 3-orthogonal coordinate system is identified by ${}^3g_{rs}(\tau, \vec{\sigma}) = 0$ for $r \neq s$ and ${}^3g_{rr} = \phi^2 \exp(\sum_{\vec{a}=1}^2 \gamma_{r\vec{a}} r_{\vec{a}})$. Then, one imposes the gauge fixing constraints $\xi^r(\tau, \vec{\sigma}) - \sigma^r \approx 0$ as a way of identifying this system of 3-coordinates with a conventional origin of the diffeomorphism group manifold.

¹³This should be compared to the Yang-Mills theory in case of a trivial principal bundle, where the corresponding variables are defined by a path integral over the original canonical variables [22,1].

2) The boundary conditions of point 1) require that the leaves of the foliations associated with the admissible 3+1 splittings of space-time must tend to Minkowski space-like hyperplanes asymptotically orthogonal to the ADM 4-momentum in a direction-independent way. This property is concretely enforced by using a technique introduced by Dirac [16] for the selection of space-times admitting asymptotically flat 4-coordinates at spatial infinity.¹⁴

3) The super-hamiltonian constraint is the generator of the gauge transformations connecting different admissible 3+1 splittings of space-time and *has nothing to do with the temporal evolution.*

4) As shown by DeWitt [25], the weakly vanishing ADM Dirac Hamiltonian has to be modified with a suitable surface term in order that functional derivatives, Poisson brackets and Hamilton equations be mathematically well-defined in such non-compact space-times. This fact, in conjunction with the points 1), 2), 3) above, entails that *there is an effective evolution in the mathematical time τ* which parametrizes the leaves of the foliation associated with any 3+1 splitting. Such evolution is ruled by the weak *ADM energy* [19,26], i.e. by a non-vanishing Hamiltonian which exists also in the reduced phase space. This is the *rest-frame instant form of metric gravity* [19]. Each gauge fixing creates a realization of the reduced phase space and the *weak ADM energy* is a functional of only the DO of that gauge. Then, the DO themselves (as any other function of them) satisfy the Hamilton equations $\dot{r}_{\bar{a}}^G = \{r_{\bar{a}}^G, E_{\text{ADM}}\}^*$, $\dot{\pi}_{\bar{a}}^G = \{\pi_{\bar{a}}^G, E_{\text{ADM}}\}^*$, where E_{ADM} is intended as the restriction of the weak *ADM Energy* to the reduced phase space and where the $\{\cdot, \cdot\}^*$ are Dirac Brackets.

5) When matter is present in this family of space-times, switching off Newton's constant ($G \mapsto 0$) yields the description of matter in Minkowski space-time foliated with the space-like hyper-planes orthogonal to the total matter 4-momentum (Wigner hyper-planes intrinsically defined by matter isolated system). In this way one gets the rest-frame instant form of dynamics reachable from parametrized Minkowski theories [1]. Incidentally, this is the first example of consistent deparametrization of general relativity in which the ADM Poincaré group tends to the Poincaré group of the isolated matter system.

These space-times are a *counterexample to the frozen time argument* based on the widespread opinion (see for instance Refs. [27]) that the Hamiltonian approach to general relativity is not fruitful, because it leads to a reduced phase space, which is a *frozen* space without evolution. For instance Belot and Earman [27] draw ontological conclusions about the absence of real (temporal) change in general relativity from the circumstance that, in

¹⁴Dirac's method brings to an enlargement of ADM canonical metric gravity with non-vanishing ADM Poincaré charges. Such space-times admit preferred asymptotic inertial observers, interpretable as fixed stars (the standard for measuring rotations). Such non-Machian properties allow to merge the standard model of elementary particles in general relativity with all the (gravitational and non-gravitational) fields belonging to the same function space (suitable weighted Sobolev spaces). Besides the existence of a realization of the Poincaré group, only one additional property is required: namely that the space-like hyper-surfaces admit an involution [23] allowing the definition of a generalized Fourier transform with its associated concepts of positive and negative energy. This disproves the claimed impossibility of defining particles in curved space-times [24].

spatially compact models of general relativity, the Hamiltonian temporal evolution boils down to a mere gauge transformation and is, therefore, physically meaningless. Instead in the previous space-times there is neither a frozen reduced phase space nor a Wheeler-DeWitt interpretation based on some local concept of time like in compact space-times. Therefore, *our gauge-invariant approach to general relativity is perfectly adequate to accommodate objective temporal change.*

In the previous Hamiltonian context there are the tools for completing Stachel's suggestion and exploiting the old proposal advanced by Bergmann and Komar [14] for an intrinsic labeling of space-time points by means of the eigenvalues of the Weyl tensor. Its four invariant scalar eigenvalues $\Lambda_W^{(k)}(\tau, \vec{\sigma})$, $k = 1, \dots, 4$, written in Petrov compressed notations, are $\Lambda_W^{(1)} = \text{Tr}({}^4C {}^4g {}^4C {}^4g)$, $\Lambda_W^{(2)} = \text{Tr}({}^4C {}^4g {}^4C {}^4\epsilon)$, $\Lambda_W^{(3)} = \text{Tr}({}^4C {}^4g {}^4C {}^4g {}^4C {}^4g)$, $\Lambda_W^{(4)} = \text{Tr}({}^4C {}^4g {}^4C {}^4g {}^4C {}^4\epsilon)$, where 4C is the Weyl tensor, 4g the metric, and ${}^4\epsilon$ the Levi-Civita totally anti-symmetric tensor. Bergman and Komar [14,28,29] proposed that we build a set of (off-shell) *invariant pseudo-coordinates* for the point-events of space-time as four suitable functions of the $\Lambda_W^{(k)}$'s, $\bar{\sigma}^{\bar{A}}(\sigma) = F^{\bar{A}}[\Lambda_W^{(k)}[{}^4g(\sigma), \partial^4g(\sigma)]]$, ($\bar{A} = 1, 2, \dots, 4$). Indeed, under the hypothesis of no space-time symmetries, we would be tempted (like Stachel) to use the $F^{\bar{A}}[\Lambda_W^{(k)}]$ as individuating fields to *label the points of space-time*, at least locally. Of course, since they are invariant functionals, the $F^{\bar{A}}[\Lambda_W^{(k)}]$'s are quantities invariant under passive diffeomorphisms (PDIQ), therefore, as such, they do not define a coordinate chart for the atlas of the mathematical Riemannian 4-manifold M^4 in the usual sense (hence the name of *pseudo-coordinates* and the superior bar used in $F^{\bar{A}}$). Moreover, the tetradic 4-metric which can be built by means of the intrinsic pseudo-coordinates is a formal object invariant under passive diffeomorphisms that does not satisfy Einstein's equations (but possibly much more complex derived equations).

The procedure of point identification starts from the fact that, within the Hamiltonian approach, Bergmann and Komar [14] proved the fundamental result that the Weyl eigenvalues $\Lambda_W^{(A)}$, once re-expressed as functionals of the Dirac (i.e. ADM) canonical variables, *do not depend on the lapse and shift functions* but only on the 3-metric and its conjugate canonical momentum, $\Lambda_W^{(k)}[{}^4g(\tau, \vec{\sigma}), \partial^4g(\tau, \vec{\sigma})] = \tilde{\Lambda}_W^{(k)}[{}^3g(\tau, \vec{\sigma}), {}^3\Pi(\tau, \vec{\sigma})]$. This result is crucial since it entails that just the *intrinsic pseudo-coordinates* $\bar{\sigma}^{\bar{A}}$ can be exploited as natural and peculiar *coordinate gauge conditions* in the canonical reduction procedure. In a completely fixed (either *off-* or *on-shell*) gauge, both the four intrinsic *pseudo-coordinates* and the ten *tetradic* components of the metric field become gauge dependent functions of the four DO of that gauge. For the Weyl scalars in particular we can write $\Lambda_W^{(k)}(\tau, \vec{\sigma})|_G = \tilde{\Lambda}_W^{(k)}[{}^3g(\tau, \vec{\sigma}), {}^3\Pi(\tau, \vec{\sigma})]|_G = \Lambda_G^{(k)}[r_{\bar{a}}^G(\tau, \vec{\sigma}), \pi_{\bar{a}}^G(\tau, \vec{\sigma})]$, where $|_G$ denotes the specific gauge. Conversely, by the inverse function theorem, in each gauge, the DO of that gauge can be expressed as functions of the 4 eigenvalues restricted to that gauge, $\Lambda_W^{(k)}(\tau, \vec{\sigma})|_G$.

Bergmann-Komar proposal can be utilized in constructing a peculiar *gauge-fixing to the super-hamiltonian and super-momentum constraints* in the canonical reduction of general relativity in the following way: after having selected a *completely arbitrary mathematical coordinate system* $\sigma^A \equiv [\tau, \sigma^a]$ adapted to the Σ_τ surfaces, one chooses *as physical individuating fields* four suitable functions $F^{\bar{A}}[\Lambda_W^{(k)}(\tau, \vec{\sigma})]$, and expresses them as functionals $\tilde{F}^{\bar{A}}$ of the ADM variables $F^{\bar{A}}[\Lambda_W^{(k)}(\tau, \vec{\sigma})] = F^{\bar{A}}[\tilde{\Lambda}_W^{(k)}[{}^3g(\tau, \vec{\sigma}), {}^3\Pi(\tau, \vec{\sigma})]] = \tilde{F}^{\bar{A}}[{}^3g(\tau, \vec{\sigma}), {}^3\Pi(\tau, \vec{\sigma})]$.

The space-time points, *mathematically individuated* by the quadruples of real numbers σ^A , become now *physically individuated point-events* through the imposition of the following gauge fixings to the four secondary constraints

$$\bar{\chi}^A(\tau, \vec{\sigma}) \stackrel{def}{=} \sigma^A - \bar{\sigma}^{\bar{A}}(\tau, \vec{\sigma}) = \sigma^A - F^{\bar{A}}[\tilde{\Lambda}_W^{(k)}[{}^3g(\tau, \vec{\sigma}), {}^3\Pi(\tau, \vec{\sigma})]] \approx 0. \quad (0.4)$$

Then, following the standard procedure a completely fixed Hamiltonian gauge, say G , is defined. This will be a correct gauge fixing provided the functions $F^{\bar{A}}[\tilde{\Lambda}_W^{(k)}(\tau, \vec{\sigma})]$ are chosen so that the $\bar{\chi}^A(\tau, \vec{\sigma})$'s satisfy the *orbit conditions* $\det|\{\bar{\chi}^A(\tau, \vec{\sigma}), \tilde{\mathcal{H}}^B(\tau, \vec{\sigma}')\}| \neq 0$, where $\tilde{\mathcal{H}}^B(\tau, \vec{\sigma}) = (\tilde{\mathcal{H}}(\tau, \vec{\sigma}); {}^3\tilde{\mathcal{H}}^r(\tau, \vec{\sigma})) \approx 0$ are the super-hamiltonian and super-momentum constraints of Eqs.(0.1). These conditions enforce the Lorentz signature on Eq.(0.4), namely the requirement that $F^{\bar{\tau}}$ be a *time* variable, and imply that *the $F^{\bar{A}}$'s are not DO*. The above gauge fixings allow in turn the determination of the four Hamiltonian gauge variables $\xi^r(\tau, \vec{\sigma})$, $\pi_\phi(\tau, \vec{\sigma})$ of Eqs.(0.3). Then, their time constancy induces the further gauge fixings $\bar{\psi}^A(\tau, \vec{\sigma}) \approx 0$ for the determination of the remaining gauge variables, i.e., the lapse and shift functions in terms of the DO in that gauge as

$$\begin{aligned} \dot{\bar{\chi}}^A(\tau, \vec{\sigma}) &= \frac{\partial \bar{\chi}^A(\tau, \vec{\sigma})}{\partial \tau} + \{\bar{\sigma}^{\bar{A}}(\tau, \vec{\sigma}), \bar{H}_D\} = \delta^{A\tau} + \\ &+ \int d^3\sigma_1 \left[N(\tau, \vec{\sigma}_1) \{\sigma^{\bar{A}}(\tau, \vec{\sigma}), \mathcal{H}(\tau, \vec{\sigma}_1)\} + N_r(\tau, \vec{\sigma}_1) \{\sigma^{\bar{A}}(\tau, \vec{\sigma}), \mathcal{H}^r(\tau, \vec{\sigma}_1)\} \right] = \\ &= \dot{\bar{\psi}}^A(\tau, \vec{\sigma}) \approx 0. \end{aligned} \quad (0.5)$$

Finally, $\dot{\bar{\psi}}^A(\tau, \vec{\sigma}) \approx 0$ determines the Dirac multipliers $\lambda^A(\tau, \vec{\sigma})$.

In conclusion, the gauge fixings (0.4) (*which break general covariance*) constitute the crucial bridge that transforms the *intrinsic pseudo-coordinates* into *true physical individuating coordinates*. As a matter of fact, after going to Dirac brackets, the point-events individuation is enforced in the form of the *identity*

$$\sigma^A \equiv \bar{\sigma}^{\bar{A}} = \tilde{F}_G^{\bar{A}}[r_{\bar{a}}^G(\tau, \vec{\sigma}), \pi_{\bar{a}}^G(\tau, \vec{\sigma})] = F^{\bar{A}}[\tilde{\Lambda}_W^{(k)}(\tau, \vec{\sigma})]|_G. \quad (0.6)$$

In this *physical 4-coordinate grid*, the 4-metric, as well as other fundamental physical entities, like e.g. the space-time interval ds^2 with its associated causal structure, and the lapse and shift functions, depend entirely on the DO in that gauge. Only on the solutions of Einstein's equations the completely fixed gauge G is equivalent to the fixation of a definite 4-coordinate system σ_G^A . The gauge fixing (0.4) ensures that *on-shell* one gets $\sigma^A = \sigma_G^A$. In this way we get a physical 4-coordinate grid on the mathematical 4-manifold M^4 dynamically determined by tensors over M^4 with a rule which is invariant under $PDiff M^4$ but such that the functional form of the map $\sigma^A \mapsto$ *physical 4-coordinates* depends on the complete chosen gauge G . This gauge-fixing makes the *invariant pseudo-coordinates* into effective *individuating fields* by forcing them to be *numerically identical* with ordinary coordinates: in this way the individuating fields turn the *mathematical* points of space-time into *physical point-events*. What really individuates space-time points physically are the very *degrees of freedom of the gravitational field*. As a consequence, one can advance the *ontological* claim that - physically - Einstein's vacuum space-time is literally *identified* with the autonomous

physical degrees of freedom of the gravitational field, while the specific functional form of the *invariant pseudo-coordinates* matches these latter into the manifold's points. The introduction of matter has the effect of modifying the Riemann and Weyl tensors, namely the curvature of the 4-dimensional substratum, and to allow *measuring* the gravitational field in a geometric way for instance through effects like the geodesic deviation equation. It is important to emphasize, however, that the addition of *matter* does not modify the construction leading to the individuation of point-events, rather it makes it *conceptually more appealing*.

The gauge fixings (0.4), (0.5) induce a *coordinate-dependent non-commutative Poisson bracket structure* upon the *physical point-events* of space-time by means of the associated Dirac brackets implying Eqs.(0.6). More exactly, on-shell, each coordinate system gets a well defined non-commutative structure determined by the associated functions $\tilde{F}_G^{\bar{A}}(r_{\bar{a}}^G, \pi_{\bar{a}}^G)$, for which we have $\{\tilde{F}_G^{\bar{A}}(r_{\bar{a}}^G(\tau, \vec{\sigma}), \pi_{\bar{a}}^G(\tau, \vec{\sigma})), \tilde{F}_G^{\bar{B}}(r_{\bar{a}}^G(\tau, \vec{\sigma}_1), \pi_{\bar{a}}^G(\tau, \vec{\sigma}_1))\}^* \neq 0$. The physical implications of this circumstance might deserve some attention in view of the quantization of general relativity.

After this solution of the problem of the identification of the point-events let us clarify the concept of *Bergmann's observable* (BO) [30]. Bergmann's definition has various facets, namely a *configurational* side having to do with invariance under *passive* diffeomorphisms, an Hamiltonian side having to do with Dirac's concept of observable, and the property of *predictability* which is entangled with both sides. According to Bergmann, (his) *observables* are passive diffeomorphisms invariant quantities (PDIQ) "which can be predicted uniquely from initial data", or "quantities that are invariant under a coordinate transformation that leaves the initial data unchanged". Bergmann says in addition that they are further required to be gauge invariant, a statement that can only be interpreted as implying that Bergmann's observables are simultaneously DO. Yet, he offers no explicit demonstration of the compatibility of this bundle of statements. The clarification of this entanglement leads to the proposal of a *main conjecture* asserting: i) *the existence of special Dirac's observables which are also Bergmann's observables*, as well as to ii) *the existence of gauge variables that are coordinate independent* (namely they behave like the tetradiac scalar fields of the Newman-Penrose formalism [31]).

The Hamiltonian approach also allows to deduce something new concerning the *overall role of gravitational and gauge degrees of freedom*. Indeed, the distinction between gauge variables and DO provided by the Shanmugadhasan transformation (0.3), conjoined with the circumstance that the Hamiltonian point of view brings naturally to a re-reading of geometrical features in terms of the traditional concept of *force*, leads to a by-product which should be added to the traditional wisdom of the equivalence principle asserting the local impossibility of distinguishing gravitational from inertial effects. Actually, the isolation of the gauge arbitrariness from the true intrinsic degrees of freedom of the gravitational field is instrumental to understand and visualize which aspects of the local effects, showing themselves on test matter, have a *genuine gravitational origin* and which aspects depend solely upon the choice of the (local) reference frame and could therefore even be named *inertial* in analogy with their non-relativistic Newtonian counterparts. Indeed, two main differences characterize the issue of *inertial effects* in general relativity with respect to the non-relativistic situation: the existence of *autonomous degrees of freedom* of the gravitational

field independently of the presence of matter sources, on the one hand, and the *local nature of the general-relativistic reference systems*, on the other. Although the very definition of *inertial forces* (and of *gravitational force* in general) is rather arbitrary in general relativity, it appears natural to characterize first of all as genuine gravitational effects those which are directly correlated to the DO, while the gauge variables appear to be correlated to the general relativistic counterparts of Newtonian inertial effects. Another aspect of the Hamiltonian connection "*gauge variables - inertial effects*" is related to the 3+1 splitting of space-time required for the canonical formalism. Each splitting is associated with a foliation of space-time whose leaves are *Cauchy simultaneity* space-like hyper-surfaces. While the field of unit normals to these surfaces identifies a *surface-forming congruence of time-like observers*, the field of the evolution vectors identifies a *rotating congruence of time-like observers*. Since a variation of the gauge variables modifies the foliation, the identification of the two congruences of time-like observers is connected to the fixation of the gauge, namely, on-shell, to the choice of 4-coordinates. Then a variation of gauge variables also modifies the inertial effects.

It is clear by now that a complete gauge fixing within canonical gravity has the following implications: i) the choice of a unique 3+1 splitting with its associated foliation; ii) the choice of well-defined congruences of time-like observers; iii) the *on-shell* choice of a unique 4-coordinate system. In physical terms this set of choices amount to choosing a *network of intertwined and synchronized local laboratories made up with test matter* (obviously up to a coherent choice of chrono-geometric standards). This interpretation shows that, unlike in ordinary gauge theories where the gauge variables are *inessential* degrees of freedom, in general relativity they describe generalized inertial effects.

The only weakness of the previous distinction is that the separation of the two autonomous degrees of freedom of the gravitational field from the gauge variables is, as yet, a coordinate (i.e. gauge) - dependent concept. The known examples of pairs of conjugate DO are neither coordinate-independent (they are not PDIQ) nor tensors. Bergmann asserts that the only known method (at the time) to build BO is based on the existence of Bergmann-Komar invariant pseudo-coordinates. A possible starting point to attack the problem of the connection of DO with BO seems to be a Hamiltonian re-formulation of the Newman-Penrose formalism [31] (it contains only PDIQ) employing Hamiltonian null-tetrads carried by the time-like observers of the congruence orthogonal to the admissible space-like hyper-surfaces. This is the source of the quoted *main conjecture* that special Darboux bases for canonical gravity should exist in which the inertial effects (gauge variables) are described by PDIQ while the autonomous degrees of freedom (DO) are *also* BO. Note that, since Newman-Penrose PDIQ are tetradic quantities, the validity of the conjecture would also eliminate the existing difference between the observables for the gravitational field and the observables for matter, built usually by means of the tetrads associated to some time-like observer. Furthermore, this would also provide a starting point for defining a metrology in general relativity in a generally covariant way¹⁵, replacing the empirical metrology [32] used

¹⁵Recall that this is the main conceptual difference from the non-dynamical metrology of special relativity

till now. It would also enable to replace by dynamical matter the *test matter* of the axiomatic approach [33] to measurement theory. This would constitute an important advance, if we recall that all of the presentations of gravitational waves and gravito-magnetism are till now coordinate-dependent. Moreover, since no-one is able to solve the super-hamiltonian constraint, it would be interesting to see how it could be expressed in such a canonical basis. After all, Ashtekar's approach started from a canonical transformation!

In Ref. [8] there is also a suggestion of how the physical individuation of space-time points, introduced at the conceptual level, could in principle be implemented with *a well-defined empirical procedure, an experimental set-up and protocol for positioning and orientation* based on the technology of the Global Positioning System. This suggestion closes the *coordinative circuit* of general relativity correlating the theoretical construction with an empirical definition of space-time.

In conclusion the rest-frame instant form of metric and tetrad gravity identifies a class of space-times where it is possible to find an answer to all the interpretation problems of general relativity. In them it is also possible to define a Hamiltonian linearization in a completely fixed non-harmonic 3-orthogonal gauge (the 3-metric is diagonal), which identifies a class of linearized post-Minkowskian vacuum Einstein space-times corresponding to *background-independent gravitational waves* [34] (see the talk of De Pietri at this Conference). The effect of the addition of matter (a relativistic perfect fluid) is now under investigation. In presence of a perfect fluid this type of background-independent linearization will make possible to define a *weak field fast motion* approximation, to find the form of the action-at-a-distance Newton and gravito-magnetic potentials and of the Dirac observable (i.e. tidal) - fluid interactions in this 4-coordinate system, without never making post-Newtonian expansions, and finally to find the relativistic quadrupole emission formula.

In generally covariant theories, the necessity of a physical identification of point-events and the chrono-geometrical aspect of the gravitational field (which teaches causality to all the other fields) put the (graviton-like) physical degrees of freedom of the gravitational field on a different level with respect to photons, gluons... This seems to be in total contrast with all the formulations on a background (like perturbative field theory and string theory). As a consequence another important motivation for looking for a canonical basis in which the gauge variables are coordinate-independent and the DO are also BO, is to try to define a new quantization scheme (respecting relativistic causality) for canonical gravity, hopefully in a Fock space and not in inequivalent Hilbert space like it happens in loop quantum gravity. In a paper in preparation [35] this new quantization scheme is defined and applied to get relativistic and non-relativistic quantum mechanics in non-inertial frames in absence of gravity (as an attempt to describe inertial effects in a framework where they are no genuine tidal, i.e. DO, effects).

Till now there are two (nearly always inequivalent) families of quantization schemes for systems with first class constraints:

i) first quantize all the canonical variables in a non physical Hilbert space and then make the reduction with respect to the gauge group arriving at the physical Hilbert space (usually a quotient); in all the approaches (BRST, geometric, algebraic and refined quantizations, deformations,...) the big problem is how to determine the physical scalar product;

ii) first reduce and then quantize; here the problem is that usually the classical reduced phase space is a highly topologically non trivial manifold.

The idea behind the new quantization scheme is to arrive directly to the physical Hilbert space by quantizing only the DO of the system and treating the gauge variables as *c-numbers* (like *time* in the time-dependent Schroedinger equation; the gauge momenta become derivatives with respect to the gauge variables, like the energy is replaced by the time derivative). In canonical gravity this scheme would make sense only if the gauge variables are coordinate-independent. There will be as many coupled Schroedinger equations as gauge variables (plus eventually one with the canonical Hamiltonian, when it is not vanishing) and the wave function will depend on as many *times* (besides the standard one) as gauge variables. Every line in this parameter space will correspond to a gauge of the classical theory. If there is an ordering such that the quantum constraints obtained with this prescription (no ordering problem for the gauge variables!) satisfy a commutator algebra with the constraints on the left of the structure functions, then the coupled Schroedinger equations will be formally integrable, the physical Schroedinger scalar product (induced by the Schroedinger equations, i.e. by the constraints) will not depend on the *times* (gauge independence) and the propagation from an initial set of times to a final one will not depend on the path in the parameter space joining these two sets. Many topological properties of the classical reduced phase space will be hidden in the properties in large of the parameter space (the new quantization scheme is only a local approximation). The first quantization of this type was obtained many years ago, in the framework of relativistic particle mechanics with first class constraints [36], with the quantization of the two-body Droz-Vincent-Todorov-Komar model with an instantaneous action-at-a-distance potential: i) the two gauge variables are the two times of the two particles; ii) the quantization gives two coupled Klein-Gordon equations; iii) in turn these equations led to the identification of four different physical scalar products, one for each branch of the mass spectrum (for non-equal masses).

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